ON SPATIAL GAS FLOWS WITH DEGENERATE HODOGRAPHS

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The papers [1,2,3,4] investigated gas flows whose hodograph which degenerates into a self-similar manifold or into a manifold of degree smaller by one than the number of independent variables. The papers [1, 2,5] investigate double waves in the case of potential flows. In the present note consideration is given to double waves without the assumption that the flow is potential. The flows [1,2,5] are obtained as a special case.

1. As is known, the equations of spatial motion of a polytropic gas for the adiabatic case have the form

$$\begin{aligned} &\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} + \frac{\partial u_4}{\partial x_i} = 0 \\ &\frac{\partial u_4}{\partial t} + u_k \frac{\partial u_4}{\partial x_k} + \lambda u_4 \frac{\partial u_k}{\partial x_k} = 0 \end{aligned} \qquad (i, k = 1, 2, 3) \quad \begin{pmatrix} u_4 = a^2/\lambda \\ \lambda = \gamma - 1 \end{pmatrix} \qquad (1.1)$$

where u_1, u_2, u_3 are the velocity components along the coordinate axes x_1, x_2, x_3 (summation is carried over repeated indexes); a is the velocity of sound; γ is the ratio of specific heats. In what follows, unless stated otherwise, as in the preceeding case, the indexes will take the values 1, 2.

Using a variation of the method presented in [4], we shall consider a case where the hodograph velocity degenerates into a two-dimensional surface, i.e.

$$u_3 = w(u_1, u_2), \qquad u_4 = \theta(u_1, u_2) \qquad (1.2)$$

It follows that the functions $u_1, \ldots, u_4$ lie on the same two-dimensional surfaces. Let us examine the case where those surfaces are planes, i.e.

$$x_k + a_{nk} y_n + a_{0k} = 0 \qquad (y_1 = x_3,\ y_2 = t) \tag{1.3}$$

where a_{nk} and a_{0k} are functions of u_1, u_2. Since a derivative along any direction located on the surface of an arbitrary function $f(u_1, \ldots, u_4)$ equals zero, we obtain

$$\frac{\partial f}{\partial y_n} = a_{nk} \frac{\partial f}{\partial x_k} \tag{1.4}$$

Using (1.2) and (1.4) and eliminating the functions u_3, u_4 and derivatives with respect to x_3, t, from System (1.1), we obtain

$$\begin{aligned} A_i &\equiv (u_k + a_{1k} w + a_{2k}) \frac{\partial u_i}{\partial x_k} + \theta_k \frac{\partial u_k}{\partial x_i} = 0 && \left(w_i = \frac{\partial w}{\partial u_i}\right) \\ A_3 &\equiv (u_k w_i + a_{1k} w w_i + a_{1k} \theta_i + a_{2k} w_i) \frac{\partial u_i}{\partial x_k} = 0 && \left(\theta_i = \frac{\partial \theta}{\partial u_i}\right) \\ A_4 &\equiv (u_k \theta_i + a_{1k} \theta_i w + a_{2k} \theta_i + \delta_{ik} \lambda \theta + a_{1k} \lambda \theta w_i) \frac{\partial u_i}{\partial x_k} = 0 && \begin{pmatrix} \delta_{ik} = 0, & i \neq k \\ \delta_{ik} = 1, & i = k \end{pmatrix} \end{aligned} \tag{1.5}$$

In what follows instead of System (1.5) we shall investigate an equivalent system, which is obtained in the following manner:

$$\begin{aligned} B_i &\equiv b_{i\alpha\beta} \frac{\partial u_\alpha}{\partial x_\beta} \equiv A_i = 0 \\ B_3 &\equiv b_{3\alpha\beta} \frac{\partial u_\alpha}{\partial x_\beta} \equiv A_i (w \theta_i + \lambda w_i \theta) - A_4 w = 0 \\ B_4 &\equiv b_{4\alpha\beta} \frac{\partial u_\alpha}{\partial x_\beta} \equiv A_i (w w_i + \theta_i) - A_3 w = 0 \end{aligned} \tag{1.6}$$

Since Equations (1.6) have to be satisfied for any y_n it is necessary to add to the system of Equations (1.6) the equations

$$\frac{\partial}{\partial y_n} \left(b_{i\alpha\beta} \frac{\partial u_\alpha}{\partial x_\beta} \right) = 0 \qquad (i = 1, \ldots, 4;\ \alpha, \beta, n = 1, 2) \tag{1.7}$$

Taking the partial derivative and applying relationships (1.4), we obtain

$$a_{nk} \frac{\partial}{\partial x_k} \left(b_{i\alpha\beta} \frac{\partial u_\alpha}{\partial x_\beta} \right) + b_{i\alpha\beta} \frac{\partial a_{nk}}{\partial x_\beta} \frac{\partial u_\alpha}{\partial x_k} = 0$$

Taking into account (1.6), we have

$$b_{i\alpha\beta} \frac{\partial a_{nk}}{\partial x_\beta} \frac{\partial u_\alpha}{\partial x_k} = 0 \qquad (i = 1, \ldots, 4;\ \alpha, \beta, k, n = 1, 2) \tag{1.8}$$

Taking into account that

$$\frac{\partial a_{nk}}{\partial x_\beta} = \frac{\partial a_{nk}}{\partial u_\delta}\frac{\partial u_\delta}{\partial x_\beta} \tag{1.9}$$

we obtain

$$\frac{\partial B_i}{\partial y_n} \equiv -J(-1)^{\alpha+\beta} b_{i\alpha\beta}\frac{\partial a_{n,3-\beta}}{\partial u_{3-\alpha}} \equiv -JL_{in} = 0 \qquad \begin{pmatrix} i=1,\ldots,4 \\ n=1,2 \end{pmatrix}$$

$$\left(J = \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2} - \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}\right) \tag{1.10}$$

In the future we shall assume that $J \neq 0$, as $J = 0$ reduces the flow to the simple wave type. Those flows were investigated by Ianenko[3].

We shall show that the relationships

$$\frac{\partial^s B_i}{\partial y^s_n} = 0 \qquad (i = 1, \ldots, 4;\ n = 1, 2;\ s = 2, 3, \ldots) \tag{1.11}$$

do not yield new equations. Indeed, using (1.4) and (1.6), we transform (1.11)

$$\frac{\partial^s B_i}{\partial y^s_n} = (-J)^{\frac{s+1}{2}} J_n^{\frac{s-1}{2}} L_{in} \qquad \text{for } s \text{ odd}$$

$$\frac{\partial^s B_i}{\partial y^s_n} = (-J)^{\frac{s}{2}} J_n^{\frac{s}{2}} B_i \qquad \text{for } s \text{ even}$$

$$\left(J_n = \frac{\partial a_{n1}}{\partial u_1}\frac{\partial a_{n2}}{\partial u_2} - \frac{\partial a_{n1}}{\partial u_2}\frac{\partial a_{n2}}{\partial u_1}\right)$$

Thus, the system of Equations (1.10) determines $w(u_1, u_2)$ and $\theta(u_1, u_2)$, i.e. the two-dimensional surface, into which the velocity hodograph degenerates.

2. We shall investigate the matrix, which consists of the coefficients of the system of Equations (1.6)

$$\left\| \begin{matrix} \zeta_1 & \zeta_2 - \theta_2 & \theta_2 & 0 \\ 0 & \theta_1 & \zeta_1 - \theta_1 & \zeta_2 \\ \lambda w_1 \theta \eta_1 + c_{11} & \lambda w_1 \theta \eta_2 + c_{12} & \lambda w_2 \theta \eta_1 + c_{21} & \lambda w_2 \theta \eta_2 + c_{22} \\ \eta_1 \theta_1 & \eta_2 \theta_1 & \eta_1 \theta_2 & \eta_2 \theta_2 \end{matrix} \right\| \tag{2.1}$$

where

$$\eta_k = a_{2k} + u_k + ww_k + \theta_k + \theta_k, \quad \zeta_k = a_{1k}w + a_{2k} + u_k + \theta_k$$

$$c_{kk} = w\theta_k^{\,2} - \lambda w\theta(1 + w_k^{\,2})$$

$$c_{km} = w\theta_k\theta_m + \lambda\theta(w_k\theta_m - w_m\theta_k) - \lambda ww_kw_m\theta \quad (k \neq m) \tag{2.2}$$

Depending on the rank of (2.1) we will obtain various flows. The case $r = 4$ leads to u_i = const. Let $r = 2$. Then obviously the solution will be

$$\eta_k = 0, \qquad \zeta_k = 0 \tag{2.3}$$

In that case we will arrive at a system of two equations tor the hodograph surface, which were initially obtained by Ryzhov [5].

Indeed, from (2.2) and (2.3) we obtain

$$a_{1k} = w_k, \qquad a_{2k} = u_k - ww_k - \theta_k \tag{2.4}$$

In that case, when the rank of the matrix (2.1) is equal to two, the system (1.10) will contain only four independent equations. When (2.4) is substituted into Equations (1.10) two of them will become identities, while the remaining two will yield the desired equations.

In the most general case the requirement that the rank of the matrix (2.1) shall be equal to two leads to four algebraic equations for the unknowns η_k and ζ_k. Determining the roots of this system, for example

$$\eta_k = ww_k + \frac{w}{\nu}[(-1)^{3-k}\theta_{3-k} + \theta_k\omega], \qquad \zeta_k = 0$$

$$\left(\nu = (-1)^{3-\alpha} w_\alpha\theta_{3-\alpha}, \ \omega^2 = -(1 + w_\alpha w_\alpha) + \tfrac{1}{\lambda\theta}(\theta_\alpha\theta_\alpha + \nu^2)\right) \tag{2.5}$$

we find a_{nk}. Substituting a_{nk} into (1.10), we obtain the system of equations for the functions $w(u_1, u_2)$ and $\theta(u_1, u_2)$.

3. Let the rank of the matrix (2.1) be equal to three. Then the system of Equations (1.6) will be dependent. Let us arrange the equations so as to make the first three equations independent. Then the System (1.10), or the equivalent, System (1.8), will contain two dependent equations with index $i = 4$.

Let us investigate three equations of the System (1.6) together with any Equation (1.8). Then the determinant of 4th order will be equal to zero, i.e.

$$D_1 b_{i1\beta}\frac{\partial a_{n1}}{\partial x_\beta} - D_2 b_{i1\beta}\frac{\partial a_{n2}}{\partial x_\beta} + D_3 b_{i2\beta}\frac{\partial a_{n1}}{\partial x_\beta} - D_4 b_{i2\beta}\frac{\partial a_{n2}}{\partial x_\beta} = 0 \qquad \begin{pmatrix} i = 1, & 2, & 3 \\ \beta, & n = 1, & 2 \end{pmatrix} \tag{3.1}$$

where D_i are determinants of the 3rd order, obtained from matrix (2.1) minus the last line, by means of delating the i-column. Otherwise u_i = const. From (3.1) we obtain

$$D_1 b_{i11} - D_2 b_{i12} + D_3 b_{i21} - D_4 b_{i22} \equiv 0 \qquad (i = 1, 2, 3) \tag{3.2}$$

Using identities (3.2), we rewrite System (3.1)

$$(D_1 b_{i11} + D_3 b_{i21}) \left(\frac{\partial a_{n1}}{\partial x_1} - \frac{\partial a_{n2}}{\partial x_2} \right) + (D_1 b_{i12} + D_3 b_{i22}) \frac{\partial a_{n1}}{\partial x_2} +$$
$$+ (-D_2 b_{i11} - D_4 b_{i21}) \frac{\partial a_{n2}}{\partial x_1} = 0 \qquad (i = 1, 2, 3;\ n = 1, 2) \tag{3.3}$$

We investigate this system for n = const with respect to

$$\left(\frac{\partial a_{n1}}{\partial x_1} - \frac{\partial a_{n2}}{\partial x_2} \right), \quad \frac{\partial a_{n1}}{\partial x_2}, \quad \frac{\partial a_{n2}}{\partial x_1}$$

We shall find its determinant Δ by computing

$$\Delta = -(D_1 D_4 - D_2 D_3)^2 \tag{3.4}$$

On the other hand, from System (1.6) which contains the first three equations, we obtain

$$J = \psi^2 (D_1 D_4 - D_2 D_3) \tag{3.5}$$

where ψ is an arbitrary function of u_1 and u_2.

Hence, if $\Delta = 0$, we obtain $J = 0$, i.e. the propagation of a simple wave; if $\Delta \neq 0$, then

$$\frac{\partial a_{n1}}{\partial x_1} - \frac{\partial a_{n2}}{\partial x_2} = 0, \qquad \frac{\partial a_{n1}}{\partial x_2} = 0, \qquad \frac{\partial a_{n2}}{\partial x_1} = 0 \tag{3.6}$$

In that case a two-parameter family of two-dimensional manifolds will intersect along a common straight line.

4. Now, let the rank of matrix (2.1) be equal to unity. It is easily established that this is possible only for

$$\zeta_k = 0, \qquad \theta_k = 0 \tag{4.1}$$

We shall investigate this case. From (2.2) we obtain

$$a_{2k} = -u_k - a_{1k} w, \qquad \eta_k = w (w_k - a_{1k}) \tag{4.2}$$

Then the System (1.10) will yield two independent equations

$$(-1)^{\alpha+\beta} (\delta_{\alpha\beta} + a_{1\beta} w_\alpha) \frac{\partial a_{1, 3-\beta}}{\partial u_{3-\alpha}} = 0, \qquad a_{1\alpha} w_\alpha + 1 = 0 \tag{4.3}$$

Substituting the value a_{12} from the second equation into the first, we determine a_{11} in terms of w

$$a_{11} = \frac{-w_1 w_{22} + w_2 w_{12} + w_2 D}{w_2^2 w_{11} - 2w_1 w_2 w_{12} + w_1^2 w_{22}} \qquad \left(w_{km} = \frac{\partial^2 w}{\partial u_k \partial u_m},\ D^2 = w_{12}^2 - w_{11} w_{22} \right) \tag{4.4}$$

For the a_{nk} selected in this manner only one independent linear equation of System (1.6) with two desired functions u_1 and u_2 has to be satisfied, which is easily accomplished. If we assume, for example, that u_2 is an arbitrary function of x_1 and x_2, then for u_1 we obtain the linear inhomogeneous equation

$$w_2 \frac{\partial u_1}{\partial x_1} - w_1 \frac{\partial u_1}{\partial x_2} = f(x_1, x_2, u_1) \tag{4.5}$$

which is easily integrated.

Using the values a_{nk} determined as functions of u_1 and u_2, and also the equations of the two-dimensional surface (1.3), we find the flow in the physical space.

5. We shall investigate the special case of steady gas flow, i.e. $a_{2k} = 0$. From Bernoulli's equation we obtain

$$\theta_k = -u_k - w w_k \tag{5.1}$$

Since $a_{2k} = 0$, it follows, that $\eta_k = 0$, i.e. in this case the last equation of (1.6) becomes an identity.

The requirement that the rank of matrix (2.1) be equal to two, leads to two independent equations for ζ_k

$$\begin{gathered} c_{12}\zeta_1^2 - c_{11}\zeta_1\zeta_2 + (c_{11}\theta_2 - c_{12}\theta_1 - c_{21}\theta_1)\zeta_1 + c_{11}\theta_1\zeta_2 = 0 \\ c_{22}\zeta_1^2 - c_{21}\zeta_1\zeta_2 - c_{22}\theta_1\zeta_1 + c_{11}\theta_2\zeta_2 = 0 \end{gathered} \tag{5.2}$$

The simplest solution $\zeta_k = 0$ leads to the equation of the hodograph surface, obtained by Nikol'skii [2].

The other solutions of Equations (5.2) have the form

$$\zeta_k = \theta_k + \frac{u_k \nu + (-1)^{3-k} w \theta_{3-k}}{\nu - w\omega} \tag{5.3}$$

where the notation is the same as in (2.5) with consideration of (5.1). The expression for ω^2 may be simplified

$$\omega^2 = (M^2 \sin^2 \tau - 1)(1 + w_\alpha w_\alpha) \tag{5.4}$$

where M is the Mach number, τ is the angle between the gas velocity and the normal, to the surface of the hodograph at the corresponding point. From (5.4) it follows that only supersonic flows are possible.

Determining a_{1k} from (5.3) and substituting it into (1.10) we obtain a system of two quasi-linear equations of the 2nd order for the surface of the hodograph.

The case when the rank of the basic system is equal to three does not yield new flows. Indeed, from Section 3 it follows, that for $\Delta = 0$ we obtain the flows of simple type wave, while $\Delta \neq 0$ applies to conical flow.

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